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Characters of permutation summands

Xiangyong Wang*

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

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Abstract

A permutation lattice for a finite group G over the ring A of integers in a number field is a free A-module with a finite A-basis which is permuted by G; direct summands of these, as AG-modules, are called permutation summands for G over A. The virtual characters are studied for these lattices through an induction theorem on virtual characters over the maximal unramified extension field of the rational p-adic numbers. © 1998 Elsevier Science B.V. All rights reserved.

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Let G be a finite group and A the ring of integers in a number field K. An AG-lattice is called a *permutation lattice* if it has an A-basis, necessarily finite, which is permuted by the action of G. It will be called a *permutation summand* (for G over A), if it is a direct summand, as an AG-module, of a permutation lattice. The Grothendieck ring $\Omega_A(G)$ of the category of all permutation summands for G over A has been studied in [11].

We are interested in surveying the characters of permutation summands of G over A. We know (from [11, (2.4)] or (3.1) below) that such characters are always $\boldsymbol{Q}$-valued, no matter what K is. Thus, we are interested in the image of the map $\varphi : \Omega_A(G) \to \bar{R}_{\boldsymbol{Q}}(G)$ which sends a lattice L to the K-character φ_L of $K \otimes_A L$ in the ring of $\boldsymbol{Q}$-valued characters of G.

The image of φ always has finite index in $\bar{R}_{\boldsymbol{Q}}(G)$, by Artin induction, and clearly grows with A. In the case $A = \boldsymbol{Z}$ its study is mainly concerned with Schur index questions. When A is big enough, the image of φ must only depend on the group structure of G: describing how is our main concern. The quaternion group $\boldsymbol{Q}_8$ of order 8 will play a special role, because of the nature of induction theorems over local fields.

For each prime p, let $\boldsymbol{Q}_p^{\mathrm{nr}}$ be the maximal unramified extension of the p-adic complete field $\boldsymbol{Q}_p$, i.e. is obtained from $\boldsymbol{Q}_p$ by adjoining all the roots of unity of order

*Correspondence address: 1699 Ontario Dr. 7, Sunnyvale, CA 94087, USA. E-mail: xywang@cs.toronto.edu.

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prime to p. Call a character of G *p'-linear* if it is one-dimensional with values roots of unity of order prime to p.

Induction Theorem. *Every $\mathbf{Q}_p^{\mathrm{nr}}$-character of a finite group G is a $\mathbf{Z}$-linear combination of induced characters* $\mathrm{ind}_H^G\,\phi$, *where either*

(i) *ϕ is a p'-linear character, or*

(ii) *$p = 2$ and ϕ is the product of a $2'$-linear character with a $\mathbf{Q}_2^{\mathrm{nr}}$-character μ of H such that $H/\ker\mu \simeq \mathbf{Q}_8$, and μ is the inflation of the unique faithful irreducible $\mathbf{Q}_2^{\mathrm{nr}}$-character of $\mathbf{Q}_8$.*

This result, which is proved in Section 1, seems not to be explicit in the literature, though it is related to the Main Theorem of [5] which can be deduced from it in the same way that Brauer induction implies that $\mathbf{Q}(\zeta_{|G|})$ is a splitting field for G [9].

Letting $R_{\mathbf{Q}_2^{\mathrm{nr}}}(G)$ be the ring of characters of $\mathbf{Q}_2^{\mathrm{nr}}$-representations of G, we define $\mathscr{R}(G)$ to be the quotient of $R_{\mathbf{Q}_2^{\mathrm{nr}}}(G)$ by the subgroup generated by induced characters $\mathrm{ind}_H^G\phi$ of $2'$-linear characters ϕ. Observe that scalar extension gives a map $\bar{R}_{\mathbf{Q}}(G) \to R_{\mathbf{Q}_2^{\mathrm{nr}}}(G)$, because $\mathbf{Q}_2^{\mathrm{nr}}$ has trivial Brauer group [10].

Main Theorem. *The image of $\varphi : \Omega_A(G) \to \bar{R}_{\mathbf{Q}}(G)$ is always contained in the kernel of the composite map $\bar{R}_{\mathbf{Q}}(G) \to R_{\mathbf{Q}_2^{\mathrm{nr}}}(G) \to \mathscr{R}(G)$. If A is big enough this containment is an equality.*

This will be proved in Section 3, with preparations in Section 2 concerning its analogue over the completions of K. It reduces the characterization of the image of φ to the problem of determining when $\chi \in R_{\mathbf{Q}_2^{\mathrm{nr}}}(G)$ represents zero in the quotient $\mathscr{R}(G)$. This question is addressed in Section 4, where it is, in particular, reduced to 2-groups.

1. Proof of Induction Theorem

We proceed by induction on the group order $|G|$. By the induction theorem of Witt–Berman [3, (21.6)], for $\mathbf{Q}_p^{\mathrm{nr}}$, we may assume that G is a $(\mathbf{Q}_p^{\mathrm{nr}}, q)$-elementary group $\langle x \rangle \rtimes Q$. The argument now depends on whether q and p are equal or not.

Case 1: $q = p$. $(\mathbf{Q}_p^{\mathrm{nr}}, p)$-elementary groups are elementary $\langle x \rangle \times P$, because $\mathbf{Q}_p^{\mathrm{nr}}$ contains all p'-roots of unity and thus triviality of $\mathrm{Gal}\,(\mathbf{Q}_p^{\mathrm{nr}}(\zeta_{|x|})/\mathbf{Q}_p^{\mathrm{nr}})$ forces a trivial action of P on $\langle x \rangle$. To complete the proof in this case we state two lemmas whose proofs are given at the end of this section.

Lemma 1.1. *Each $\mathbf{Q}_p^{\mathrm{nr}}$-irreducible character χ of $G_1 \times G_2$ is a product of $\mathbf{Q}_p^{\mathrm{nr}}$-irreducibles χ_1 of G_1 with χ_2 of G_2, whenever* $\gcd(|G_1|, |G_2|) = 1$.

Lemma 1.2. *If G is a p-group then $R_{\mathbf{Q}_p^{\mathrm{nr}}}(G)$ is spanned by*

(a) *permutation characters; and*

(b) *if $p = 2$, all induced characters of the form $\mathrm{ind}_H^G\,\mu$ with μ inflating the unique faithful $\mathbf{Q}_2^{\mathrm{nr}}$-irreducible character θ of $H/\ker\mu = \mathbf{Q}_8$.*

Now if χ is an irreducible $\mathbf{Q}_p^{\mathrm{nr}}$-character of $G = \langle x\rangle \times P$, then χ is a product of irreducible $\mathbf{Q}_p^{\mathrm{nr}}$-characters χ_1 of $\langle x\rangle$ and χ_2 of the p-group P, by Lemma 1.1; χ_1 is necessarily p'-linear, and χ_2, by Lemma 1.2 above, is a $\mathbf{Z}$-linear combination of induced characters $\mathrm{ind}_{P_i}^P\,\psi$, where ψ is either trivial or a μ. It follows that χ is a $\mathbf{Z}$-linear combination of characters $\chi_1\cdot\mathrm{ind}_{P_i}^P\,\psi = \mathrm{ind}_{\langle x\rangle\rtimes P_i}^G(\mathrm{res}\chi_1\cdot\psi)$. So the Induction Theorem is established for $(\mathbf{Q}_p^{\mathrm{nr}}, p)$-elementary groups.

Case 2: $q \neq p$. Using the decomposition $x = x_{p'}x_p$ of elements of G into p, p'-parts, we can write the $(\mathbf{Q}_p^{\mathrm{nr}}, q)$-elementary group as $\langle x\rangle \rtimes \mathbf{Q} = (\langle x_{p'}\rangle \times \langle x_p\rangle)\rtimes \mathbf{Q} = \langle x_{p'}\rangle \times (\langle x_p\rangle \rtimes \mathbf{Q})$ since $\mathbf{Q}$ must act trivially on $\langle x_{p'}\rangle$. By Lemma 1.1 the $\langle x_{p'}\rangle$ does not matter so the Induction Theorem follows from

Proposition 1.3. *Suppose $G = C \rtimes D$ with a cyclic p-group C and a p'-group D. Then every irreducible $\mathbf{Q}_p^{\mathrm{nr}}$-character χ of G is a $\mathbf{Z}$-linear combination of induced characters $\mathrm{ind}_H^G\,\phi$ of p'-linear characters ϕ.*

Proof. Proceeding by induction on $|G|$, we may assume χ is faithful.

If C is trivial, the lemma follows from Brauer's Induction Theorem as $\mathbf{Q}_p^{\mathrm{nr}}$ contains all p'th roots of unity. Let $|C| = p^n$, $n \geq 1$. The kernel of the homomorphism $D \to \mathrm{Aut}\,C$ is $C_D(C)$, and the image of D is necessarily a p'-subgroup of $\mathrm{Aut}\,C$, hence is cyclic.

Denote $C_D(C)$ by D_0 and let $H = C \times D_0$. Then H is normal and $G/H \simeq D/D_0$ is a cyclic p'-group. Letting η be a $\mathbf{Q}_p^{\mathrm{nr}}$-constituent of $\mathrm{res}_H\chi$, then $\eta = \xi\mu$ with $\xi \in \mathrm{Irr}_{\mathbf{Q}_p^{\mathrm{nr}}}(C)$ and $\mu \in \mathrm{Irr}_{\mathbf{Q}_p^{\mathrm{nr}}}(D_0)$ by Lemma 1.1. As χ is a $\mathbf{Q}_p^{\mathrm{nr}}$-constituent of $\mathrm{ind}_H^G\,\eta$ by Frobenius reciprocity and $\ker\xi$ is normal in G, $\ker\xi$ acts trivially on $\mathrm{ind}_H^G\,\eta$ and therefore trivially on χ. Since χ is faithful, we have $\ker\xi = 1$. Then ξ is the unique faithful $\mathbf{Q}_p^{\mathrm{nr}}$-irreducible character of C, hence its inertia group is G. Let $D_1 = I_D(\mu) = \{t \in D\colon \mu^t = \mu\}$. Then the inertia group T of $\eta = \xi\cdot\mu$ is

$$T = I_G(\eta) = I_G(\xi)\cap I_G(\mu) = G\cap(C\rtimes D_1) = C\rtimes D_1\,.$$

We delay the proof of the following lemma to the end of the section.

Lemma 1.4. *Suppose G has a self-centralizing cyclic normal subgroup C of order p^n, and let C_p be the unique subgroup of C of order p. Then*

(a) *G has a unique $\mathbf{Q}_p^{\mathrm{nr}}$-irreducible character θ on which C_p acts non-trivially. Moreover, $\theta|_C$ is the unique faithful $\mathbf{Q}_p^{\mathrm{nr}}$-irreducible character of C and has degree $p^{n-1}(p-1)$.*

(b) *If C has a complement in G, then θ is a virtual permutation character.*

Applying the above lemma to $C \rtimes (D_1/D_0)$, we obtain the unique faithful character θ. This is an extension of ξ and is virtual permutation character. Letting $\tilde{\xi}$ be the inflation of θ through $C \rtimes D_1 \to C \rtimes (D_1/D_0)$, then the $\boldsymbol{Q}_p^{\mathrm{nr}}$-character $\tilde{\xi}$ of T is an extension of ξ and is a virtual permutation character. On the other hand, since D_1/D_0 is a cyclic p'-group and $\boldsymbol{Q}_p^{\mathrm{nr}}$ contains all p'th-roots of unity, the Extension Theorem [7, (11.22)] applied to μ and $D_0 \lhd D_1$ asserts that μ has an extension $\tilde{\mu}$ in $\mathrm{Irr}_{\boldsymbol{Q}_p^{\mathrm{nr}}}(D_1)$. Denote the inflation of $\tilde{\mu}$ through $C \rtimes D_1 \to D_1$ still by $\tilde{\mu}$. Then $\tilde{\mu} \in \mathrm{Irr}_{\boldsymbol{Q}_p^{\mathrm{nr}}}(T)$ is an extension of μ. Combining the above, $\eta = \zeta \cdot \mu$ has an extension $\tilde{\xi} \cdot \tilde{\mu}$, denoted by $\tilde{\eta}$, to its inertia group T.

Frobenius reciprocity gives $\mathrm{ind}_H^T \eta = \tilde{\eta} \cdot \mathrm{ind}_H^T 1$ because $\mathrm{res}_H^T \tilde{\eta} = \eta$. Let $\mathrm{ind}_H^T 1 = \mathrm{ind}_1^{T/H} 1 = \sum_i \lambda_i$ be the decomposition into $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducibles. Since T/H is a cyclic p'-group and $\boldsymbol{Q}_p^{\mathrm{nr}}$ contains all p'th roots of unity, these λ_i are necessarily p'-linear. Products $\tilde{\eta} \cdot \lambda_i$ must be $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducible because λ_i is one-dimensional and $\tilde{\eta}$ is $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducible. Therefore,

$$\mathrm{ind}_H^T \eta = \tilde{\eta} \cdot \mathrm{ind}_H^T 1 = \sum_i \tilde{\eta} \lambda_i$$

is the decomposition of $\mathrm{ind}_H^T \eta$ into $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducibles.

Now each $\psi \in \mathrm{Irr}_{\boldsymbol{Q}_p^{\mathrm{nr}}}(T)$ with $(\eta, \mathrm{res}_H \psi) \neq 0$ is a $\boldsymbol{Q}_p^{\mathrm{nr}}$-constituent of $\mathrm{ind}_H^T \eta$ by Frobenius reciprocity and thus is one of the $\tilde{\eta}\lambda_i$ by the last paragraph. The Theorem of Clifford [7, (6.11)] applied to χ and η, gives $\chi = \mathrm{ind}_T^G \psi$ for a $\psi \in \mathrm{Irr}_{\boldsymbol{Q}_p^{\mathrm{nr}}}(T)$ with $(\eta, \mathrm{res}_H \psi) \neq 0$. Therefore,

$$\chi = \mathrm{ind}_T^G(\tilde{\eta}\lambda_i) = \mathrm{ind}_T^G(\tilde{\xi}\tilde{\mu}\lambda_i) = \mathrm{ind}_T^G(\tilde{\xi} \cdot \tilde{\mu}\lambda_i),$$

where $\tilde{\xi}$ is a virtual permutation character, λ_i is a p'-linear character and $\tilde{\mu}$ is an inflation of a $\boldsymbol{Q}_p^{\mathrm{nr}}$-character of the p'-group D_1 and thus is a $\boldsymbol{Z}$-linear combination of induced characters of p'-linear characters by Brauer's Induction Theorem. The lemma then follows from Frobenius reciprocity as in the first paragraph of Case 1. The proof of Proposition 1.3 is completed. □

Proof of Lemma 1.1. As every finite extension of $\boldsymbol{Q}_p^{\mathrm{nr}}$ has trivial Brauer group, the Wedderburn decompositions $\boldsymbol{Q}_p^{\mathrm{nr}}[G_i] \simeq \prod_j M_{n_j}(K_j^{(i)})$, for $i = 1, 2$, have fields $K_j^{(i)}$. These fields are generated by character values [2, (70.8)], hence are linearly disjoint over $\boldsymbol{Q}_p^{\mathrm{nr}}$. Thus, all $K_j^{(1)} \otimes K_{j'}^{(2)}$ are fields and

$$\boldsymbol{Q}_p^{\mathrm{nr}}[G] \simeq \boldsymbol{Q}_p^{\mathrm{nr}}[G_1] \otimes \boldsymbol{Q}_p^{\mathrm{nr}}[G_2] \simeq \prod_{j,j'} M_{n_j n_{j'}}(K_j^{(1)} \otimes K_{j'}^{(2)})$$

is the Wedderburn decomposition. The lemma follows on considering the characters of the simple components. □

Proof of Lemma 1.2. For each $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducible character ψ, write $\psi = \mathrm{ind}_H^G \mu$ so that μ is $\boldsymbol{Q}_p^{\mathrm{nr}}$-*primitive* (i.e. μ is not induced from a $\boldsymbol{Q}_p^{\mathrm{nr}}$-character of a proper subgroup of H). Then the lemma follows from the claim below applied to the character of $H/\ker \mu$ which inflates to μ, by noting that the faithful irreducible $\boldsymbol{Q}_p^{\mathrm{nr}}$-character of the cyclic group C_p is $\mathrm{ind}_1^{C_p} 1 - \mathrm{ind}_{C_p}^{C_p} 1$. □

Claim. *Suppose G is a p-group and has a faithful irreducible $\boldsymbol{Q}_p^{\mathrm{nr}}$-character χ which is $\boldsymbol{Q}_p^{\mathrm{nr}}$-primitive. Then G is either cyclic of order p, or $p = 2$ and G is the quaternion group Q_8 of order* 8.

Proof of the Claim. Let A be an abelian normal subgroup of G, and let η be an irreducible $\boldsymbol{Q}_p^{\mathrm{nr}}$-constituent of $\mathrm{res}_A^G \chi$. Then η is G-stable because χ is primitive, and χ is a constituent of $\mathrm{ind}_A^G \eta$ [7, (6.11)]. Now $\ker \eta \lhd G$, by η G-stable, so $\ker \eta$ acts trivially on $\mathrm{ind}_A^G \eta$, hence on χ. Then $\ker \eta = 1$, by χ faithful, so η is faithful on abelian group A. Thus A is cyclic.

We have just shown that every abelian normal subgroup of p-group of G is cyclic. By group theory [6, (5.4.10)] either G is cyclic or $p = 2$ and G is dihedral, semidihedral, quaternion. We now analyze χ case by case.

If G is cyclic of order p^n then the $\boldsymbol{Q}_p^{\mathrm{nr}}$-irreducible character χ on which G_p (the cyclic subgroup of order p) acts non-trivially, is unique and has degree $p^{n-1}(p-1)$. If ξ is this character of degree $p-1$ for G_p then G_p acts non-trivially on the induced character $\mathrm{ind}_{G_p}^G \xi$. So χ is a constituent of $\mathrm{ind}_{G_p}^G \xi$. Comparing degrees gives $\chi = \mathrm{ind}_{G_p}^G \xi$. Since χ is primitive, it follows that $G = G_p$ is cyclic of order p, which is the first possibility the claim names.

Thus, $p = 2$ and G has a cyclic normal group C of index 2. By Lemma 1.4(a), χ is the unique $\boldsymbol{Q}_2^{\mathrm{nr}}$-character on which C_2 acts non-trivially, and χ has degree $\frac{1}{4}|G|$. Just as in the last paragraph this implies that χ is induced from a $\boldsymbol{Q}_2^{\mathrm{nr}}$-character of a subgroup H. This H can be taken noncyclic of order 4 if G is dihedral or semidihedral, and to be quaternion of order 8 if G is quaternion. □

Proof of Lemma 1.4. (a) Since the p^nth cyclotomic polynomial is irreducible over $\boldsymbol{Q}_p^{\mathrm{nr}}$ by the Eisenstein criterion, we have

$$\boldsymbol{Q}_p^{\mathrm{nr}}[C] \simeq \boldsymbol{Q}_p^{\mathrm{nr}}\left[\frac{C}{C_p}\right] \times \boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n}).$$

Hence, $\boldsymbol{Q}_p^{\mathrm{nr}}[G] = \boldsymbol{Q}_p^{\mathrm{nr}}[C] \circ (G/C)$ can be expressed as crossed product algebras

$$\boldsymbol{Q}_p^{\mathrm{nr}}[G] \simeq \boldsymbol{Q}_p^{\mathrm{nr}}\left[\frac{G}{C_p}\right] \times \boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n}) \circ (G/C)$$

with G/C acting as a Galois group on $\boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n})$, and for some factor set in $H^2(G/C,\ \boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n})^{\times})$. Since the $\boldsymbol{Q}_p^{\mathrm{nr}}[G]$-irreducible modules on which C_p acts non-trivially are the $\boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n}) \circ (G/C)$ modules, and since every finite extension of $\boldsymbol{Q}_p^{\mathrm{nr}}$ has trivial Brauer group, it remains only to observe that $\boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n}) \circ (G/C)$ is a simple algebra with *split* factor set [8, (29.6), (29.12)]. It follows that its simple module is just $\boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n})$ with $\boldsymbol{Q}_p^{\mathrm{nr}}(\zeta_{p^n})$ acting by multiplication, and G/C by Galois action.

(b) Write $G = C \rtimes D$ and $\mathrm{ind}_D^{C_p \rtimes D} 1 = 1_{C_p \rtimes D} + \alpha$, with α a proper character; consider $\mathrm{ind}_{C_p \rtimes D}^G \alpha$. Now $\mathrm{ind}_D^G 1 = \mathrm{ind}_{C_p \rtimes D}^G 1 + \mathrm{ind}_{C_p \rtimes D}^G \alpha$ and C_p acts non-trivially on

$\mathrm{ind}_D^G 1$, trivially on $\mathrm{ind}_{C_p \rtimes D}^G 1$, hence non-trivially on $\mathrm{ind}_{C_p \rtimes D}^G \alpha$. So θ is a $\mathbf{Q}_p^{\mathrm{nr}}$-constituent of $\mathrm{ind}_{C_p \rtimes D}^G \alpha$ with $(\mathrm{ind}_{C_p \rtimes D}^G \alpha)(1) = [G : C_p \rtimes D]\,\alpha(1) = p^{\bar{n}1}.(p-1) = \theta(1)$. Hence, $\theta = \mathrm{ind}_{C_p \rtimes D}^G \alpha$ is a difference of two transitive permutation characters. $\square$

2. Local results

Let k be a finite extension field of $\mathbf{Q}_p$, and let $\mathfrak{o}$ be the integral closure of the p-adic integers $\mathbf{Z}_p$ in k. *In this section, we always assume that k contains the* $|G|_p$*th roots of unity.*

Let $\Omega_{\mathfrak{o}}(G)$ be the Grothendieck group of the category of permutation summands for G over $\mathfrak{o}$, and $R_{K_{\mathfrak{p}}}(G)$ the group generated by the characters of the representations of G over k. Mapping each lattice to its k-character, we obtain a ring homomorphism $\varphi : \Omega_{A_{\mathfrak{p}}}(G) \to R_{K_{\mathfrak{p}}}(G)$ as in the global situation.

In this section, we study this local image, via the Green correspondence in connection with the study of characters of projective $\mathfrak{o}G$-modules. Let $P_{\mathfrak{o}}(G)$ be the Grothendieck group of the category of finitely generated projective $\mathfrak{o}G$-modules, and let $e : P_{\mathfrak{o}}(G) \to R_{K_{\mathfrak{p}}}(G)$ send each projective to its k-character as usual [9].

Lemma 2.1. *The image of* $e : P_{\mathfrak{o}}(G) \to R_{K_{\mathfrak{p}}}(G)$ *is the subgroup generated by induced characters* $\mathrm{ind}_{P'}^G \lambda$ *of linear k-characters* λ *of* p'*-subgroups* P' *of* G.

Proof. It is clear that each $\mathrm{ind}_{P'}^G \lambda$ is in the image of e. By [5, Lemma 1], each character of a projective is an integral linear combination of induced characters of elementary subgroups of p'-order. Now the lemma follows from Brauer Induction applied to p'-order elementary subgroups. $\square$

Proposition 2.2. *The image of* $\varphi : \Omega_{A_{\mathfrak{p}}}(G) \to R_{K_{\mathfrak{p}}}(G)$ *is the subgroup generated by induced characters* $\mathrm{ind}_H^G \phi$ *of* p'*-linear characters* ϕ *of subgroups* H *of* G.

Proof. Since p'-linear characters are clearly the characters of permutation summands over $\mathfrak{o}$, it suffices to exhibit a $\mathbf{Z}$-basis of $\Omega_{A_{\mathfrak{p}}}(G)$ and then show that their characters are sums of induced characters $\mathrm{ind}_H^G \phi$.

The Grothendieck group $\Omega_{A_{\mathfrak{p}}}(G)$ has a $\mathbf{Z}$-basis, by Krull–Schmidt and vertex theory, parameterized by pairs (P, V), where P is a p-subgroup (determined up to conjugacy) and V is an indecomposable permutation summand $\mathfrak{o}G$-lattice with vertex P. The Green correspondent $f_P(V)$ is an indecomposable $\mathfrak{o}[N_G(P)]$-module with vertex P. Since P acts trivially on $f_P(V)$ by [4, Section 81B], $f_P(V)$ can be considered as an indecomposable projective $\mathfrak{o}[N_G(P)/P]$-module M will give an indecomposable $\mathfrak{o}[N_G(P)]$-module of vertex P by inflation [4, (81.15) (iii)]. Then $\mathrm{ind}_{N_G(P)}^G(\inf M)$, parameterized by (P, M), is a second $\mathbf{Z}$-basis of $\Omega_{A_{\mathfrak{p}}}(G)$ because the Green relations

$$\mathrm{ind}_{N_G(P)}^G(\inf M) = V \oplus V', \quad \mathrm{vtx}(V') \subsetneq P$$

provide a transition matrix which is upper triangular with 1's on the main diagonal.

Denote $N_G(P)/P$ by $\bar{N}_G(P)$ for simplicity. The character χ_M, in the image of $e: P_o(\bar{N}_G(P)) \to R_o(\bar{N}_G(P))$, is expressible as $\chi_M = \sum_i n_i \operatorname{ind}_{P_i'}^{\bar{N}_G(P)} \lambda_i$ by Lemma 2.1. Thus its inflation is $\inf \chi_M = \sum_i n_i \operatorname{ind}_{H_i}^{N_G(P)} \phi_i$, where each H_i is the preimage of P_i', ϕ_i is the inflation of λ_i and thus is a p'-linear character of H_i. The images of the basis $\{\operatorname{ind}_{N_G(P)}^G (\inf M) | (P, M)\}$ in $R_{K_p}(G)$ are then $\operatorname{ind}_{N_G(P)}^G \inf \chi_M = \sum_i n_i \operatorname{ind}_{H_i}^G \phi_i$ as required. □

Since p'-linear characters of G are realizable over $\boldsymbol{Q}_p^{\mathrm{nr}}$, we get a map $\varphi : \Omega_o(G) \to R_{\boldsymbol{Q}_2^{\mathrm{nr}}}(G)$.

Corollary 2.3. *The homomorphisms $\varphi : \Omega_o(G) \to R_{\boldsymbol{Q}_p^{\mathrm{nr}}}(G)$ are surjective for odd primes p. If $p = 2$, the cokernel $\mathscr{R}(G)$ is annihilated by 2.*

Proof. The first assertion follows from Proposition 2.2 and the Induction Theorem. For the second, $\mathscr{R}(G)$ is generated by characters of form $\operatorname{ind}_H^G(\psi \cdot \mu)$, by (ii) of the Induction Theorem, where ψ is a 2′-linear character and μ is the inflation of the unique faithful irreducible character θ of $\boldsymbol{Q}_8$, so it suffices to observe that $2\theta = \operatorname{ind}_1^{Q_8} 1 - \operatorname{ind}_{C_4}^{Q_8} 1$ is a virtual permutation character. □

3. Proof of Main Theorem

For completeness' sake we include a different proof for the following proposition [11, (2.4)].

Proposition 3.1. *Given a permutation summand L of G over A, let φ_L denote the character of $K \otimes_A L$. Then the value $\varphi_L(x)$ is in $\boldsymbol{Z}$ for each element x in G.*

Proof. We may assume G is cyclic of order n, generated by x. For each prime divisor p of n, we can write $G = E \times P$, where P is a p-group and E is of order $n_{p'}$ prime to p.
Since $\gcd\{n_{p'} : p|n\} = 1$ implies $\bigcap_{p|n} \boldsymbol{Q}(\zeta_{n_{p'}}) = \boldsymbol{Q}$, our result follows from

Claim. *$\varphi_L(x)$ is a sum of $n_{p'}$th roots of unity for each p.*

For the purpose of proving this claim, we may enlarge K by adjoining $n_{p'}$th roots of unity and by completing at some prime $\mathfrak{p}$ above p, i.e. we may replace $A \subset K$ by $\mathfrak{o} \subset k$ in the notation of Section 2. We may also assume that L is an indecomposable permutation summand of G over $\mathfrak{o}$.

If D is the vertex of L, then L is a direct summand of $\operatorname{ind}_D^G(\mathfrak{o})$ by [4, Section 81B]. Since D is normal in G, we may consider $\operatorname{ind}_D^G(\mathfrak{o})$ and L as $\mathfrak{o}[G/D]$-modules which are then projective. If $D \subsetneq P$, then xD is p-singular in G/D, hence $\varphi_L(x) = 0$ [9, Theorem 36]. Otherwise, xD has order $n_{p'}$, so $\varphi_L(x)$ is a sum of $n_{p'}$th roots of unity. This proves the Claim and Proposition 3.1. □

The images of $\varphi : \Omega_{A_\mathfrak{p}}(G) \to R_{Q_p^{nr}}(G)$ are characterized in Corollary 2.3 on all local rings $\mathfrak{o}$, whenever $\mathfrak{o}$ contains $|G|_{p'}$th roots of unity. To prove the Main Theorem for the global ring A, we use the technique of gluing permutation summand lattices over the completions $A_\mathfrak{p}$ at all $\mathfrak{p}$ to form a permutation summand lattice over A.

Lemma 3.2. *Given a KG-module V, and for each $\mathfrak{p}$ above a rational prime divisor of $|G|$, let there be given a permutation summand $Y(\mathfrak{p})$ of G over $A_\mathfrak{p}$, such that $K_\mathfrak{p} \otimes_{A_\mathfrak{p}} Y(\mathfrak{p}) = K_\mathfrak{p} V$. Then there exists a permutation summand L of G over A, such that*

$$KL = V, \qquad A_\mathfrak{p} \otimes_A L \simeq Y(\mathfrak{p}) \quad \textit{for all such } \mathfrak{p}.$$

Proof. Let M be a G-stable A-submodule in V such that $KM = V$. Denote by $\mathscr{S}$ the set of prime ideals of A lying above rational prime divisors of the group order $|G|$. Define

$$L = V \cap \left\{ \bigcap_{\mathfrak{p} \in \mathscr{S}} Y(\mathfrak{p}) \right\} \cap \left\{ \bigcap_{\mathfrak{p} \notin \mathscr{S}} (A_\mathfrak{p} \otimes_A M) \right\},$$

where the intersection is taken over all prime ideals $\mathfrak{p}$ of A. Then $KL = V$, and $A_\mathfrak{p} \otimes_A L \simeq Y(\mathfrak{p})$ for $\mathfrak{p} \in \mathscr{S}$, follow from [8, (5.3) (ii)]. L is a permutation summand of G over A by Lemmas 1 and 2 in [1], on replacing $\boldsymbol{Z}$ by A. $\square$

Proof of Main Theorem. For each prime ideal $\mathfrak{p}$ of A above a prime number p which divides the group order $|G|$, we consider the $\mathfrak{p}$-adic completion $K_\mathfrak{p}$ with the ring of integers $A_\mathfrak{p}$. Let k be an extension field of $K_\mathfrak{p}$ containing $|G|_{p'}$th roots of unity, and let $\mathfrak{o}$ be the integer ring of k. Then the first part of the Main Theorem follows from Proposition 2.2 and the commutative diagram

$$\begin{tikzcd} \Omega_A(G) \arrow[r, "\varphi"] \arrow[d, "\mathfrak{o}\otimes_A"'] & \bar{R}_Q(G) \arrow[d] \\ \Omega_\mathfrak{o}(G) \arrow[r, "\varphi"] & R_{Q_p^{nr}}(G) \end{tikzcd}$$

For the second part, we call a number field K *big enough* (with respect to G) if it satisfies the following two conditions:

(1) The completion $K_\mathfrak{p}$ contains $|G|_{p'}$th roots of unity for each $\mathfrak{p}$ above a prime divisor p of $|G|$.

(2) All rational valued characters are realizable over K.

The field $\boldsymbol{Q}(\zeta_{|G|})$, for instance, is one example of a big enough field. Alternatively, we can arrange that $K/\boldsymbol{Q}$ is unramified at all prime divisors of $|G|$ by the theorem of Grunwald–Wang.

The second part of the theorem for big enough K amounts to: given a virtual character χ in the kernel of $\bar{R}_Q(G) \to \mathscr{R}(G)$, we want to construct a (virtual) permutation summand x in $\Omega_A(G)$ such that the K-character of x is χ. Since K is big enough, we have $\chi \in \bar{R}_Q(G) \subset R_K(G)$, and the local fields $K_\mathfrak{p}$ satisfy the requirement of Section 2.

By Proposition 2.2, it follows that for each prime $\mathfrak{p}$ of K above a prime divisor p of $|G|$, there exists $x(\mathfrak{p}) \in \Omega_{A_\mathfrak{p}}(G)$ so that $\varphi_{x(\mathfrak{p})} = \chi$ holds in $R_{Q_p^{nr}}(G)$. For $p = 2$ we need to use our hypothesis that χ represents 0 in $\mathscr{R}(G)$ here.

Next, for each $\mathfrak{p} \in S$, equal to the set of primes of K above rational prime divisors of $|G|$, write $x(\mathfrak{p}) = [M_1] - [M_2]$ as a difference of permutation summands for G over $A_\mathfrak{p}$. Then $M_2 \oplus M_2'' \simeq A_\mathfrak{p}[S(\mathfrak{p})]$ for some G-set $S(\mathfrak{p})$ and $A_\mathfrak{p}G$-lattice M_2'', so, setting $X(\mathfrak{p}) = M_1 \oplus M_2''$, we have $x(\mathfrak{p}) = [X(\mathfrak{p})] - [A_\mathfrak{p}[S(\mathfrak{p})]]$ in $\Omega_{A_\mathfrak{p}}(G)$.

Then $S = \dot{\bigcup}_{\mathfrak{p} \in \mathscr{S}} S(\mathfrak{p})$ is a G-set, so on setting $Y(\mathfrak{p}) = X(\mathfrak{p}) \oplus A_\mathfrak{p}[S \setminus S(\mathfrak{p})]$, we have $x(\mathfrak{p}) = (Y(\mathfrak{p})) - (A_\mathfrak{p}[S])$ in $\Omega_{A_\mathfrak{p}}(G)$ for each $\mathfrak{p} \in \mathscr{S}$. Let the character of $A_\mathfrak{p}[S]$ be φ_S, which is determined by the G-set S and is independent of $\mathfrak{p}$. Since $x(\mathfrak{p})$ has character χ by construction, the character of $K_\mathfrak{p} \otimes_{A_\mathfrak{p}} Y(\mathfrak{p})$ is $\chi + \varphi_S$. It follows that the virtual character $\chi + \varphi_S \in R_K(G)$ is indeed a K-character afforded by a KG-module V [9, Proposition 33]. Applying now Lemma 3.2 to V, $Y(\mathfrak{p})$, we have a permutation summand L for G over A, such that $\varphi_L = \chi + \varphi_S$. Setting $x = [L] - [A[S]]$ in $\Omega_A(G)$, then $\varphi_x = \varphi_L - \varphi_S = \chi$ as desired. $\square$

4. About $\mathscr{R}(G)$

Given a character χ in $R_{Q_2^{nr}}(G)$, we want to determine whether it represents zero in the quotient $\mathscr{R}(G)$. The first proposition reduces this problem to 2-elementary groups and then to 2-groups.

Proposition 4.1. (a) $\mathscr{R}(G) \xrightarrow{\text{res}} \otimes_E \mathscr{R}(E)$ *is injective, where E ranges over 2-elementary subgroups of G.*

(b) *If $E = C \times P$ is 2-elementary then $\mathscr{R}(E) = R_{Q_2^{nr}}(C) \otimes \mathscr{R}(P)$.*

Proof. (a) Assume the result is false and that G is a counterexample of least order. By Solomon's induction Theorem [3, (15.10)] there is a relation

$$1_G = \sum_H n_H \operatorname{ind}_H^G(1_H)$$

with H ranging over hyperelementary subgroups of G. If G is not hyperelementary then multiplying this relation with $\chi \in \mathscr{R}_{Q_2^{nr}}(G)$ representing an element of the kernel in (a) gives a contradiction. Thus, G is hyperelementary.

Next we show that $\mathscr{R}(G) = 0$ if G is p-hyperelementary with $p \neq 2$. Write $G = C \rtimes P$ with C cyclic p' and P a p-group, and write $C = T \times T'$ with T a cyclic 2-group and T' of odd order. Since $\mathrm{Aut}(T)$ is a 2-group, P acts trivially on T hence $G = T \times G_1$ with $G_1 = T' \rtimes P$ of odd order. Now $R_{Q_2^{nr}}(G) = R_{Q_2^{nr}}(T) \otimes R_{Q_2^{nr}}(G_1)$, by

Lemma 1.1. Here $R_{\mathbf{Q}_2^{\mathrm{nr}}}(T)$ is spanned by permutation characters while $R_{\mathbf{Q}_2^{\mathrm{nr}}}(G_1)$ is spanned by induced characters of 2′-linear characters by Brauer's induction theorem (since $\mathbf{Q}_2^{\mathrm{nr}}$ contains $|G_1|$th roots of unity). It follows that $\mathscr{R}(G) = 0$.

It follows that G is 2-hyperelementary but not 2-elementary. By the argument of the first paragraph it suffices to establish a relation

$$(*) \qquad 1_G = \sum_{H \neq G} n_H \operatorname{ind}_L^G(\phi_H)$$

in $R_{\mathbf{Q}_2^{\mathrm{nr}}}(G)$, where each ϕ_H is a 2′-linear character of a *proper* subgroup of G.

Write $G = C \rtimes P$ with C cyclic of odd order and P a 2-group; by hypothesis P acts non-trivially on C. It suffices to prove $(*)$ for some quotient of G, as it then follows for G by inflation. This permits us to replace G by any quotient which is not 2-elementary.

Since P must act non-trivially on some primary component of C we may assume C is a cyclic q-group with $q \neq 2$. Then P acts non-trivially on C/C^p so we may suppose C has order p. Factoring by the kernel of the action of P on C we may assume, since $\operatorname{Aut}(C)$ is cyclic, that P is a cyclic 2-group of order $m > 1$ which acts faithfully on C.

For such a $G = C \rtimes P$ it is easy to determine the $\mathbf{Q}_2^{\mathrm{nr}}$-irreducible characters. In particular, if S is a set of representatives of the action of P on the 2′-linear characters ϕ of C, then $\{\operatorname{ind}_C^G \phi : \phi \in S\}$ consists of $(q-1)/m$ different $\mathbf{Q}_2^{\mathrm{nr}}$-irreducible characters of G. Each of these is a constituent of $\operatorname{ind}_p^G 1$ so we get a relation

$$\operatorname{ind}_p^G 1 = 1_G + \sum_{\phi \in S} \operatorname{ind}_C^G \phi$$

on comparing degrees. This proves $(*)$, hence (a).

(b) Denote by $S_2(E)$ the subgroup of $R_{\mathbf{Q}_2^{\mathrm{nr}}}(E)$ generated by induced characters of form $\operatorname{ind}_{E'}^E \phi$, where ϕ is 2′-linear $\mathbf{Q}_2^{\mathrm{nr}}$-character of E'. We take the definition of $\mathscr{R}$, tensored by $R_{\mathbf{Q}_2^{\mathrm{nr}}}(C)$ in the top row, to form a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R_{\mathbf{Q}_2^{\mathrm{nr}}}(C) \otimes S_2(P) \arrow[r] \arrow[d] & R_{\mathbf{Q}_2^{\mathrm{nr}}}(C) \otimes R_{\mathbf{Q}_2^{\mathrm{nr}}}(P) \arrow[r] \arrow[d, "\simeq"'] & R_{\mathbf{Q}_2^{\mathrm{nr}}}(C) \otimes \mathscr{R}(P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_2(E) \arrow[r] & R_{\mathbf{Q}_2^{\mathrm{nr}}}(E) \arrow[r] & \mathscr{R}(E) \arrow[r] & 0
\end{tikzcd}$$

The middle vertical isomorphism $\alpha \otimes \beta \mapsto (\inf_{E \to C} \alpha)(\inf_{E \to P} \beta)$ is that of Lemma 1.1 and this induces the other vertical maps. It then suffices to show that the left vertical is onto.

Take a generator $\operatorname{ind}_{E'}^E \phi$ of $S_2(E)$, with ϕ 2′-linear character of E'. Writing $E' = C' \times P'$ we have $\phi = \inf_{E' \to C'} \alpha$ with $\alpha \in R_{\mathbf{Q}_2^{\mathrm{nr}}}(C')$. Hence, $\operatorname{ind}_{E'}^E \phi = \operatorname{ind}_{C \times P'}^{C \times P}(\operatorname{ind}_{C' \times P'}^{C \times P'} \phi) = \operatorname{ind}_{C \times P'}^{C \times P}(\inf_{C \times P' \to C} \operatorname{ind}_{C'}^C \alpha) = \operatorname{ind}_{C \times P'}^{C \times P}(\inf_{C \times P \to C} \operatorname{ind}_{C'}^C \alpha){\downarrow}_{C \times P'} = (\inf_{C \times P \to C} \operatorname{ind}_{C'}^C \alpha)\ (\operatorname{ind}_{C \times P'}^{C \times P} 1) = (\inf_{E \to C} \operatorname{ind}_{C'}^C \alpha)\ (\inf_{E \to P} \operatorname{ind}_{P'}^P 1)$ is in the image of $R_{\mathbf{Q}_2^{\mathrm{nr}}}(C) \otimes S_2(P)$, as required.

The result of (b) means that $\chi \in R_{Q_2^{nr}}(G)$ can be written $\chi = \sum_\alpha \alpha\chi_\alpha$ with unique $\chi_\alpha \in R_{Q_2^{nr}}(P)$, where α varies through the 2′-linear characters of C. Thus, $\chi = 0$ in $\mathscr{R}(G)$ if and only if $\chi_\alpha = 0$ in $\mathscr{R}(P)$ for all α. $\square$

It remains to study $\mathscr{R}(G)$ when G is a 2-group, which will be the case from now on. We know, from (2.3), that $\mathscr{R}(G)$ is a vector space over the field $\boldsymbol{F}_2$, and, from (the claim in the proof of) (1.2), that $\mathscr{R}(G)$ is spanned by those irreducible $\boldsymbol{Q}_2^{nr}$-characters χ which are not virtual permutation characters, i.e. for which $\chi = \mathrm{ind}_{H_1}^G(\mathrm{inf}_{H_1 \to H_1/H_0}\theta)$ where H_1/H_0 is quaternion of order 8 and θ is its unique faithful irreducible $\boldsymbol{Q}_2^{nr}$-character. This parameterization of generators χ is not very efficient; a better one is given by

Proposition 4.2. *Let χ be a $\boldsymbol{Q}_2^{nr}$-irreducible character of a 2-group G which is not a virtual permutation character. Then G has a subgroup H so that $N_G(H)/H$ is a quaternion group of order 2^n for some $n \geq 3$ and*

$$\chi = \mathrm{ind}_{N_G(H)}^G(\mathrm{inf}_{N_G(H) \to N_G(H)/H}\,\theta),$$

where θ is the unique faithful irreducible $\boldsymbol{Q}_2^{nr}$-character of $N_G(H)/H$.

Proof. We know that $\chi = \mathrm{ind}_{H_1}^G(\mathrm{inf}_{H_1 \to H_1/H_0}\theta)$ where $H_1/H_0 = \boldsymbol{Q}$ is a quaternion group of order 2^n for some $n \geq 3$ and θ is the unique faithful irreducible $\boldsymbol{Q}_2^{nr}$-character of $\boldsymbol{Q}$. Choose such an expression with n maximal; we must show that $H_1 = N_G(H_0)$.

If this were false then $\boldsymbol{Q}$ would have index 2 in some subgroup K of $N_G(H_0)/H_0$. Since then $\chi = \mathrm{ind}_{\hat{K}}^G(\mathrm{inf}_{\hat{K} \to K}(\mathrm{ind}_Q^K\theta))$ with $K = \hat{K}/H_0$, our result follows from the claim below: for (a) or (b) contradicts our hypothesis on χ and (c) contradicts the maximality of n (as $\hat{K}$ can replace H_1).

Claim. *One of the following happens*:

(a) $\mathrm{ind}_Q^K\theta$ *is a virtual permutation character,*

(b) $\mathrm{ind}_Q^K\theta$ *is reducible,*

(c) K *is a quaternion group.*

To prove this claim we write $\boldsymbol{Q} = \langle x, y\colon x^{2^{n-2}} = y^2,\ yxy^{-1} = x^{-1}\rangle$ and must examine the possibilities for K. Most of these will turn out to be in case (a) by the use of

Criterion. Suppose K contains an element h so that, (i) $h^2 = 1$, (ii) $K = Q \rtimes \langle h\rangle$, and (ii) h is K-conjugate to y^2h. Then $\mathrm{ind}_Q^K\theta$ is a virtual permutation character.

Indeed (i)–(iii) give enough information about the conjugacy class structure of K to calculate that $\mathrm{ind}_Q^K\theta = \mathrm{ind}_{\langle h\rangle}^K 1 - \mathrm{ind}_{\langle y^2, h\rangle}^K 1$.

Let $a \in K$ generate K/Q. Changing notation if necessary we may assume $axa^{-1} = x^r$ for some $r \equiv 1 \bmod 4$. It follows, from $a^2 \in Q$, that $r^2 \equiv 1 \bmod 2^{n-1}$.

And examining the conjugation action of K on $\langle x\rangle$ we get

$$(C_K(\langle x\rangle):\langle x\rangle) = \begin{cases} 2, & r \equiv 1 \bmod 2^{n-1}, \\ 1, & r \not\equiv 1 \bmod 2^{n-1}. \end{cases}$$

We first take care of the special case in which $C_K(\langle x\rangle)$ is cyclic of order 2^n. Then $C_K(\langle x\rangle) = \langle a, x\rangle$ so we may choose notation so $a^2 = x$. Then examining the action of y on $\langle a\rangle$ we must have $yay^{-1} = a^{-1}$ or $a^{-1}y^2$. The first of these possibilities is in case (c) and the second in case (a) by the Criterion with $h = ay$.

In all other cases the group extension

$$1 \to \langle x\rangle \to \langle x, a\rangle \to \langle x, a\rangle/\langle x\rangle \to 1$$

splits. If $r \equiv 1 \bmod 2^{n-1}$ this follows from the last paragraph; otherwise, we have $n \geq 4$ and $r \equiv 1 + 2^{n-2}$ and 2^{n-1} from which 2-cohomology can be easily calculated.

Thus, $a^2 = 1$, again adjusting notation. Moreover, we now have $yay^{-1} = y^{2j}a$ with $j \in \boldsymbol{Z}/2\boldsymbol{Z}$. This follows from $yay^{-1} = x^i a$ with $i \in \boldsymbol{Z}/2^{n-1}\boldsymbol{Z}$: for squaring gives $x^{i(1+r)} = 1$, hence $i \equiv 0 \bmod 2^{n-2}$ and $x^i \in \langle y^2\rangle$.

Most of the remaining possibilities are in case (a), by the Criterion with $h = a$. Indeed this works if $j = 1$ or if $j = 0$ and $r \not\equiv 1 \bmod 2^{n-1}$.

So we may assume $j = 0$ and $r \equiv 1 \bmod 2^{n-1}$. Then $K = Q \times \langle a\rangle$ and we are in case (b): for if $\tilde{\theta} = \inf_{K \to Q} \theta$ then $\operatorname{ind}_Q^K \theta = \tilde{\theta} \operatorname{ind}_Q^K 1 = \tilde{\theta}(1 + \alpha) = \tilde{\theta} + \tilde{\theta}\alpha$ with α the non-trivial character of K/Q. $\square$

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